JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 16, 405-418 (1966)

On Scattering of Coherent Waves by a General Class of Deterministic Objects Imbedded in Random Media

YUNG MING CHEN[1]

Department of Engineering Science and Mechanics, University of Florida, Gainesville, Florida

Submitted by J. M. Richardson

1. INTRODUCTION

The study of the scattering of coherent waves by deterministic objects (single or multiple) imbedded in random media is of great practical importance in many branches of modern science and engineering concerning the phenomena of scattering of waves. In general, its importance is manifested in the following two manners. First, if one hopes to examine the case in which the medium is not exactly known but in which the probability of the medium in any one of the family of media is known, then one can determine the probability that the wave motion is any one of the associated family of wave motions and hence the mean (cohenrent) wave motion can be known. Secondly, if one hopes to examine the case in which the known medium is very complex and the determination of the associated wave motion is impractical, one may choose a random medium in which certain statistical properties of wave motion is closely related to the actual properties of the wave motion. In spite of the importance of this problem, very little has been done except the recent work by Chen [1] in which a special case, the scattering of coherent wave by a perfectly conducting sphere imbedded in a random medium, is considered.

It is our main purpose in this paper to study the scattering of coherent waves by many deterministic objects imbedded in random media. However, before we can examine the many-body problem, we must study the single-scatterer problem, and this can be done by extending the method proposed by Chen [1]. We first present the perturbation analysis for computing the expectation (or mean) value of the solution of a general stochastic linear partial differential equation. Next, we formulate the problem of scattering of waves by an object imbedded in a random medium mathematically as a boundary value problem. Upon incorporating the effect of the boundary

[1] Associate Professor.

condition into the stochastc linear partial differential equation by the method of "pseudopotential," we may calculate the coherent wave motion by utilizing the previous derived result. We have found that except in a few special cases, the coherent wave motion can be computed from the corresponding deterministic scattering problem with the mean propagation constant k replaced by an effective propagation constant $k\tilde{n}$. For illustrating the method without introducing extraneous details, two special examples are worked out explicitly in the Appendix. Finally, some remarks about extending the single-scatterer problem to the many-body problem are given.

2. Stochastic Linear Partial Differential Equation

The stochastic linear partial differential equation $L(q)\, u = g(q)$ is a family of linear partial differential equations depending upon a parameter q with values ranging over a space Ω_q in which a probability density $P(q)$ is defined. The probability density $P(q)$ determines the probability of a given value of q and consequently the corresponding linear partial differential equation of the family. If for each value of q there is determined a unique solution $u(q)$ of the equation, then $u(q)$ is a random variable with certain probability density $P_u(q)$. The expectation (or mean) value of $u(q)$, by definition, is given by

$$\langle u \rangle = \int_{\Omega_q} u(q)\, P_u(q)\, dq. \tag{1}$$

It is essential to establish the equation for $\langle u \rangle$ by perturbation method and to investigate its solution, $\langle u \rangle$. For small random deviation, $L(q)$ and $g(q)$ can be expanded into power series of ϵ, where ϵ is a small parameter measuring the randomness in $L(q)$ and $g(q)$, and the linear equation becomes

$$[L_0 + \epsilon L_1(q) + \epsilon^2 L_2(q)]\, u = g_0 + \epsilon g_1(q) + \epsilon^2 g_2(q) + 0(\epsilon^3). \tag{2}$$

Let

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + 0(\epsilon^3) \tag{3}$$

where obviously

$$L_0 u_0 = g_0 \,. \tag{4}$$

If the inverse operator L_0^{-1} is well defined, it can be solved formally from Eqs. (2), (3), and (4) that

$$u_0 + \epsilon u_1 + \epsilon^2 u_2 = \epsilon u_0 L_0^{-1}(g_1 - L_1 u_0) + \epsilon^2 L_0^{-1}(g_2 - L_2 u_0 - L_1 u_1) + 0(\epsilon^3), \tag{5}$$

which in turn indicates

$$u_1 = L_0^{-1} g_1 - L_0^{-1} L_1 u_0 . \tag{6}$$

Thus, Eq. (3) can be written as

$$u = u_0 + \epsilon L_0^{-1}(g_1 - L_1 u_0) + \epsilon^2 L_0^{-1}(g_2 - L_2 u_0 - L_1 L_0^{-1} g_1 + L_1 L_0^{-1} L_1 u_0) + 0(\epsilon^3), \tag{7}$$

and its expectation value is

$$\langle u \rangle = u_0 + \epsilon L_0^{-1}(\langle g_1 \rangle - \langle L_1 \rangle\, u_0) + \epsilon^2 L_0^{-1}(\langle g_2 \rangle - \langle L_2 \rangle\, u_0$$

$$- \langle L_1 L_0^{-1} g_1 \rangle + \langle L_1 L_0^{-1} L_1 \rangle\, u_0) + 0(\epsilon^3). \tag{8}$$

To obtain the equation satisfied by $\langle u \rangle$, it is still necessary to eliminate u_0 in (8). This can be achieved by deriving the expression of u_0 in terms of $\langle u \rangle$, correct through terms of order ϵ, from (8) and substituting (4) into (8). Hence

$$\langle u \rangle = u_0 + \epsilon L_0^{-1}(\langle g_1 \rangle - \langle L_1 \rangle \langle u \rangle) + \epsilon^2 L_0^{-1}(\langle L_1 \rangle L_0^{-1} \langle g_1 \rangle + \langle g_2 \rangle$$

$$- \langle L_1 L_0^{-1} g_1 \rangle - \langle L_2 \rangle \langle u \rangle - \langle L_1 \rangle L_0^{-1} \langle L_1 \rangle \langle u \rangle$$

$$+ \langle L_1 L_0^{-1} L_1 \rangle \langle u \rangle) + 0(\epsilon^3). \tag{9}$$

By applying L_0 to both sides of the above equation, the following important equation for $\langle u \rangle$ is established:

$$[L_0 + \epsilon \langle L_1 \rangle + \epsilon^2(\langle L_2 \rangle + \langle L_1 \rangle L_0^{-1} \langle L_1 \rangle - \langle L_1 L_0^{-1} L_1 \rangle)]\, \langle u \rangle$$

$$= g_0 + \epsilon \langle g_1 \rangle + \epsilon^2(\langle g_2 \rangle + \langle L_1 \rangle L_0^{-1} \langle g_1 \rangle - \langle L_1 L_0^{-1} g_1 \rangle) + 0(\epsilon^3). \tag{10}$$

This shows a very interesting phenomenon, the coupling between the differential operators and the inhomogeneous terms forming new inhomogeneous terms. Upon examining the above derivation, one may wonder why the extra effort is spent to derive the equation satisfied by $\langle u \rangle$ and to obtain $\langle u \rangle$ by solving (10), while (8) already gives the explicit form of $\langle u \rangle$. This is because of the complex expression of (8) which does not suggest any physical interpretation of the coherent wave motion to us. On the other hand, upon inspecting and solving (10), we have a much clearer physical understanding of $\langle u \rangle$. All these will be demonstrated in Section 4 later on.

The above derivation is essentially the same as that of Chen [1]. Keller [2] has derived the special form of (10) in which $g_1 = g_2 = 0$. However, both of them fail to point out the importance of having both Eqs. (8) and (10).

3. General Formulation for Single-Scatterer Problem

The problem of scattering of a plane wave by an object imbedded in a random medium can be formulated mathematically as the following boundary value problem. For the case of opaque scatterer, the total field satisfies the following system of equations,

$$\nabla^2 u + k^2 n^2(\bar{r}, q)\, u = 0 \tag{11}$$

in the exterior of B, the boundary of the scatterer, the boundary condition

$$Z_1 \frac{\partial u}{\partial n} + Z_2 u = 0 \qquad \text{on} \qquad B, \tag{12}$$

where Z_1 and Z_2 are two complex constants, and the radiation condition

$$\lim_{|\bar{r}| \to \infty} |\, \bar{r}\, | \left(\frac{\partial u}{\partial \,|\, \bar{r}\, |} - iknu \right) = 0 \tag{13}$$

In order to use Eq. (10), we must incorporate the effect of the boundary condition (12) into the partial differential equation (11) such that the new partial differential equation with no boundary condition gives exactly the same solution as that of (11) and (12) in the exterior region of B and on B. For this purpose, we shall formally insert a symbolic operator or a "pseudopotential" (Appendix B in Huang [3], Liu and Wong [4]), L_B, as an equivalent for the boundare condition (12). This approach is justified in the theory of partial differential equation in Courant and Hilbert ([5], p. 318). In general L_B is a stochastic operator and its expression is not unique. However, it can be shown from the energy flux consideration that L_B is a deterministic operator for Eq. (12) either equal to $u = 0$ or equal to $\partial u/\partial n = 0$. The special case of a perfectly conducting or hard sphere is given in Liu and Wong [4]. Furthermore, if the object is very dense and quite lossy, L_B is approximately a deterministic operator. Now, Eqs. (11) and (12) are equivalent to

$$[\nabla^2 + L_B + k^2 n^2(\bar{r}, q)]\, u = 0. \tag{14}$$

4. Scattering of Coherent Wave by an Object Imbedded in a Random Medium

For the random media which differ slightly from the homogeneous media, $n(\bar{r}, q)$ has the following form

$$n(\bar{r}, q) = 1 + \epsilon\omega(\bar{r}, q) + 0(\epsilon^2). \tag{15}$$

If L_B is a deterministic operator, upon comparing (14) and (2) we have

$$L_0 = \nabla^2 + k^2 + L_B, \tag{16}$$

$$L_1 = 2k^2\omega(\bar{r}, q), \tag{17}$$

$$L_2 = k^2\omega^2(\bar{r}, q), \tag{18}$$

and

$$g_0 = g_1 = g_2 = 0.$$

Hence

$$\langle L_1 \rangle = 2k^2 \langle \omega \rangle \tag{19}$$

and

$$\langle L_2 \rangle = k^2 \langle \omega^2 \rangle \tag{20}$$

To simplify the problem, $\langle \omega \rangle = 0$ is assumed. Then Equation (10) yields

$$[L_0 + \epsilon^2(\langle L_2 \rangle - \langle L_1 L_0^{-1} L_1 \rangle)] \langle u \rangle = 0(\epsilon^3) \tag{21}$$

with its appropriate radiation condition. The inverse operator L_0^{-1} can be written as the integral operator

$$L_0^{-1} f(\bar{r}) = - \int [G_{in}(\bar{r}, \bar{r}') + G_s(\bar{r}, \bar{r}')] f(\bar{r}') \, d\bar{r}' \tag{22}$$

where $G_{in}(\bar{r}, \bar{r}')$ and $G_s(\bar{r}, \bar{r}')$ are the parts of Green's function of L_0 due to the incident field and the scattered field, respectively. It is well known that

$$G_{in}(\bar{r}, \bar{r}') = \frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi \mid \bar{r} - \bar{r}' \mid}. \tag{23}$$

Upon utilizing (17) and (22), we obtain

$$\langle L_1 L_0^{-1} L_1 \rangle \langle u \rangle = -4k^4 \int [G_{in}(\bar{r}, \bar{r}') + G_s(\bar{r}, \bar{r}')] \, C(\bar{r}, \bar{r}') \langle u(\bar{r}', q) \rangle \, d\bar{r}' \tag{24}$$

where

$$C(\bar{r}, \bar{r}') = \langle \omega(\bar{r}, q) \, \omega(\bar{r}', q) \rangle \tag{25}$$

is the correlation function for the random fluctuation of the medium. Now $C(\bar{r}, \bar{r}')$ is assumed to be a function of $\ell = \mid \bar{r} - \bar{r}' \mid$ only. This is appropriate when the medium is statistically homogeneous and isotropic in space.

It is also known that $G_s(\bar{r}, \bar{r}')$ is completely determined by the properties of the scatterer and hence $G_s(\bar{r}, \bar{r}')$ is quite different for each special case, nevertheless it has certain general properties. The three most important properties are given in the following:

(i) It is obvious that the radiation condition (13) insures $G_s(\bar{r}, \bar{r}') = 0(\eta^{-1})$ for $k\eta \gg 1$ regardless of the boundary condition at B, the size and the shape of the scattere, where η is the distance measured away from the boundary of the scatterer.

(ii) For the class of scatterers fitted in the eleven curvilinear three-dimensional coordinates in which the reduced wave equation is separable and for $k\bar{a} \ll 1$ ($\bar{a}$ being a typical dimension of the scatterer), $G_s(\bar{r}, \bar{r}') = 0(k\bar{a})^{\Gamma}$ where $\Gamma > 0$ and depends on the properties of the scatterer. To justify this, one has to rely on the method of eigenfunction expansion (see the content and notation of [11], Chapter 7).

Now let $\bar{r} = (\xi_1, \xi_2, \xi_3)$, $\bar{r}' = (\xi_1', \xi_2', \xi_3')$, and $\xi_1 = \text{constant} = (\bar{a}/2)$ define B, the boundary of the scatterer.

By the method of eigenfunction expansion,

$$G_{in}(\bar{r}, \bar{r}') = -4\pi\left(\frac{h_1}{h_2 h_3}\right)\rho(\xi_2', \xi_3')\sum_q \frac{\overline{W}_q(\xi_2', \xi_3')\, W_q(\xi_2, \xi_3)}{\Delta(y_{1q}, y_{2q})} \times \begin{cases} y_{1q}(k\xi_1)\, y_{2q}(k\xi_1'), & \xi_1 \leqslant \xi_1', \\ y_{1q}(k\xi_1')\, y_{2q}(k\xi_1), & \xi_1 \geqslant \xi_1', \end{cases} \tag{26}$$

and

$$G_s(\bar{r}, \bar{r}') = -4\pi\left(\frac{h_1}{h_2 h_3}\right)\rho(\xi_2', \xi_3')\sum_q A_q \frac{\overline{W}_q(\xi_2', \xi_3')\, W_q(\xi_2, \xi_3)}{\Delta(y_{1q}, y_{2q})} \times \begin{cases} y_{2q}(k\xi_1)\, y_{2q}(k\xi_1'), & \xi_1 \leqslant \xi_1' \\ y_{2q}(k\xi_1')\, y_{2q}(k\xi_1), & \xi_1 \geqslant \xi_1', \end{cases} \tag{27}$$

where the constant A_q is determined by the boundary condition (12) and

$$A_q = -\left[\frac{Z_1 y_{1q}'\left(\frac{k\bar{a}}{2}\right) + Z_2 y_{1q}\left(\frac{k\bar{a}}{2}\right)}{Z_1 y_{2q}'\left(\frac{k\bar{a}}{2}\right) + Z_2 y_{2q}\left(\frac{k\bar{a}}{2}\right)}\right]. \tag{28}$$

Since the eigenfunction $y_{1q}(k\bar{a}/2)$ and $y_{2q}(k\bar{a}/2)$ are special functions, they always can be expanded into asymptotic series of $k\bar{a}$ for $k\bar{a} \ll 1$, and hence $A_q = 0(k\bar{a})^{\Gamma}$ (that is, $G_s(\bar{r}, \bar{r}') = 0(k\bar{a})^{\Gamma}$). However, to show $\Gamma > 0$ one has to work out every special case. Instead of doing so, only the cases involving spherical scatterer are presented ([1] and Appendix) for the demonstration. After all, any smooth convex surface can be approximated to any degree of accuracy by a collection of spherical surface-elements with different radii (Fig. 1).

(iii) For the case of smooth convex scatterers and $k\bar{a} \gg 1$, $G_s(\bar{r}, \bar{r}') = 0(k\bar{a})^{\tau}$ where $\tau < 0$ and depends on the properties of the scatterer. The justification of this result relies solely on the method of "geometrical

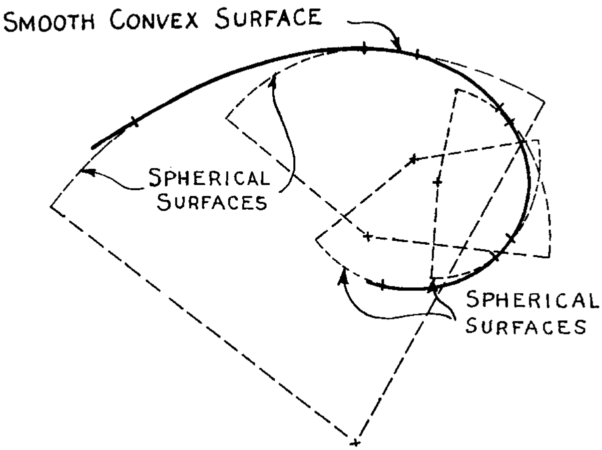

FIG. 1. The approximation of a smooth convex surface by a collection of elements of spherical surfaces is shown.

theory of diffraction" of Keller [6, 7]. Since the geometrical theory of diffraction is a constructive method of obtaining asymptotic solutions in diffraction problems with $k\bar{a} \gg 1$ and is different in every special case [8, 9], only the cases involving spherical scatterer are considered ([1] and Appendix) for the demonstration. This is sufficient, because again any smooth convex surface can be approximated to within any degree of accuracy by a collection of spherical surface-elements with different radii (Fig. 1).

Next, let us find the situations where

$$I_s = \int G_s(\bar{r}, \bar{r}')\, C(\ell)\, \langle u(\bar{r}') \rangle \, d\bar{r}' \tag{29}$$

is small enough and of order ϵ. To do so we make use of the first general property of $G_s(\bar{r}, \bar{r}')$ and obtain $I_s = 0(1/\eta)$. This indicates that I_s decreases as η increases and hence there must exist a boundary layer around the scatterer such that $I_s = 0(\epsilon)$ in the exterior of the layer (Fig. 2). The typical thickness of this layer is denoted by λ. Furthermore, if the typical dimension $\bar{a}$ of the scatterer is considered as a new perturbation parameter, from the second and the third general properties of $G_s(\bar{r}, \bar{r}')$, λ can be controlled by varying $\bar{a}$. To be more specific, the thickness of the boundary layer decreases for both decreasing and increasing of $\bar{a}$ in the cases of $k\bar{a} \ll 1$ and $k\bar{a} \gg 1$, respectively.

Since the presence of I_s is solely due to the presence of $G_s(\bar{r}, \bar{r}')$ which in turn is due to the presence of the scatterer, then the physical interpretation of the effect of I_s on $\langle u \rangle$ can only be that I_s represents the complex effects of interaction between the boundary of the scatterer and the surrounding

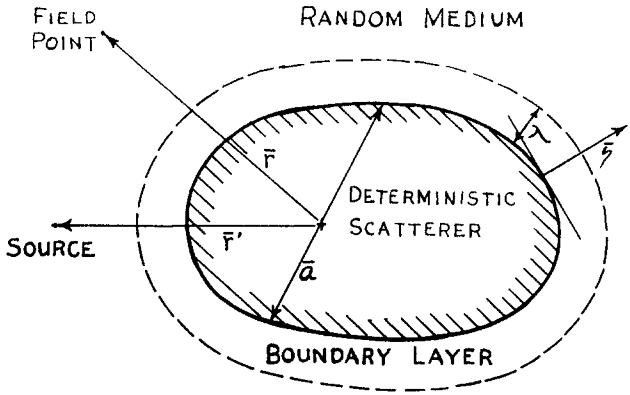

FIG. 2. The boundary layer in the exterior of the deterministic scatterer is shown.

random medium. This interaction dies down like the Coulomb force as $\bar{r}$ moves away from B and hence in the region outside the boundary layer $I_s = 0(\epsilon)$. Fortunately, in most practical problems, only the scattered far field is of great interest and there $I_s = 0(\epsilon)$.

Now, in the exterior of the boundary layer, the coherent wave $\langle u \rangle$ satisfies the differential-integral equation

$$(\nabla^2 + k^2 + L_B + \epsilon^2 k^2 \langle \omega^2 \rangle) \langle u \rangle + \epsilon^2 4k^4 \int G_{in}(\bar{r}, \bar{r}')\, C(\ell) \langle u(\bar{r}') \rangle \, d\bar{r}' = 0(\epsilon^3). \tag{30}$$

By employing the mean value theorem for the solution of the Helmholtz wave equation in Courant and Hilbert ([5], p. 288), we evaluate the integral of (30) by first integrating over the surface of a sphere of radius r centered at $\bar{r}'$. After some algebra, we obtain

$$\int G_{in}(\bar{r}, \bar{r}')\, C(\ell) \langle u(\bar{r}') \rangle \, d\bar{r}' = \frac{1}{i2k} \langle u \rangle \int_0^\infty (e^{i2k\ell} - 1)\, C(\ell)\, d\ell. \tag{31}$$

Upon substituting (31) into (30), we have

$$[\nabla^2 + L_B + k^2 \tilde{n}^2] \langle u \rangle = 0(\epsilon^3), \tag{32}$$

where

$$\tilde{n}^2 = 1 + \epsilon^2 \left[\langle \omega^2 \rangle - i2k \int_0^\infty (e^{i2k\ell} - 1)\, C(\ell) d\ell \right]. \tag{33}$$

This means that in the exterior of the boundary layer, $\langle u \rangle$ can be calculated from the equivalent deterministic scattering problem which can be obtained from the original deterministic problem ($\epsilon = 0$) by replacing k with the effect propagation constant $k\tilde{n}$.

The imaginary part of $k\tilde{n}$ is the attenuation constant of the coherent wave and it can be shown from (33) as

$$\operatorname{Im}(k\tilde{n}) = \epsilon k^2 \int_0^\infty (1 - \cos 2k\ell)\, C(\ell)\, d\ell \qquad \text{for} \qquad k \text{ real.} \tag{34}$$

Also,

$$\operatorname{Re}(k\tilde{n}) > (1 + \epsilon^2 \langle \omega^2 \rangle) \operatorname{Re}(k) \tag{35}$$

has been shown by Meecham [10] and Keller [2]. These show that the effective attenuation and the effective phase velocity of the coherent wave are increased and decreased, respectively, by the randomness of the medium. These are because of the scattering of wave increasing the distance in which it travels from one point to another.

For the case of field point, $\bar{r}$, inside the boundary layer, Eqs. (32) and (33) do not hold, and we must find other methods of obtaining $\langle u \rangle$. Fortunately, if we know u_0, the wave in the original deterministic problem, we can calculate $\langle u \rangle$ numerically from (8) as

$$\langle u \rangle = u_0 + \epsilon^2 k^2 \int G(\bar{r}, \bar{r}') \left[\langle \omega^2 \rangle\, u_0(\bar{r}') + 4k^2 \int G(\bar{r}', \bar{r}'')\, C(\ell)\, u_0(\bar{r}'')\, d\bar{r}'' \right] d\bar{r}' + 0(\epsilon^3). \tag{36}$$

Because of the complex expression of (36), we fail to give any physical interpretation in detail. This is in striking contrast with the case of $\bar{r}$ outside the boundary layer.

Finally, for illustrating the method and the result of this section without introducing extraneous details, two simple examples are worked out explicitly in the Appendix.

5. Scattering of Coherent Wave by Many Objects Imbedded in a Random Medium

When many deterministic objects are imbedded in a random medium which differs slightly from the homogeneous one, the phenomenon of multiple scattering arises, and it is of prime importance. In general, calculation of the multiple-scattering coherent wave for an arbitrary configuration of arbitrary scatterers is not possible to carry out. However, if the spacings between the scatterers are much larger than the wavelength, from the results for the single object in previous sections, we find that for the case of N objects, except in those thin boundary layers where the interactions between the boundaries of the scatterers and the surrounding random medium are important, $\langle u \rangle$ satisfies

$$(\nabla^2 + k^2 \tilde{n}^2) \langle u \rangle = 0(\epsilon^3) \tag{37}$$

in the exterior of $B_1, B_2, \ldots, B_N$, the boundaries of the scatterers 1, 2,... N, respectively, the boundary conditions

$$F_i \left(\langle u \rangle, \frac{\partial \langle u \rangle}{\partial n} \right) = 0 \quad \text{on} \quad B_i, \qquad i = 1, 2, \ldots N, \tag{38}$$

provided the corresponding L_{B_i} being deterministic or approximately deterministic. Equations (37) and (38) can be solved by utilizing the methods in Zitron and Karp [12] and Twersky [13]. Since the methods are well known, we shall not reproduce them in this paper.

6. Appendix

To illustrate our results in Section 4, we calculate the coherent wave scattered by a lossy transparent sphere imbedded in a random medium. Let the center of the sphere be the origin of a spherical coordinate system (r, θ, φ) and the incident wave be independent of φ. Then the total field u satisfies the following system of equations,

$$\nabla^2 u + k_1^2 [1 + \epsilon \omega(r, q)]^2 u = 0 \quad \text{for} \quad r \geqslant a, \tag{39}$$

where $\omega(a, q) = 0$;

$$\nabla^2 u + k_2^2 u = 0 \quad \text{for} \quad r < a, \tag{40}$$

$$u \Big|_{r=a} = \alpha u \Big|_{r=a-\delta}, \tag{41}$$

$$\frac{\partial u}{\partial r} \Big|_{r=a} = \beta \frac{\partial u}{\partial r} \Big|_{r=a-\delta}, \tag{42}$$

where δ is a small constant but $\delta > 0$, and

$$\lim_{r \to \infty} r \left[\frac{\partial u}{\partial r} - ik_1(1 + \epsilon\omega) u \right] = 0. \tag{43}$$

Without loss of generality, we shall assume that $G_s(\bar{r}, \bar{r}')$ is independent of φ and $\bar{r}'$ and $(r', 0, 0)$.

It is well known that

$$G_s(\bar{r}, \bar{r}') = -\frac{ik_1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \frac{\alpha j_n(k_2a) j_n'(k_1a) - \beta N j_n'(k_2a) j_n(k_1a)}{\alpha j_n(k_2a) h_n^{(1)\prime}(k_1a) - \beta N j_n'(k_2a) h_n^{(1)}(k_1a)} \cdot P_n(\cos\theta)\, h_n^{(1)}(k_1 r_>)\, h_n^{(1)}(k_1 r_<) \tag{44}$$

with

$$N = \left(\frac{k_2}{k_1}\right), \qquad |N| \gg 1, \qquad r_> = \max(r, r'), \qquad \text{and} \qquad r_< = \min(r, r').$$

Obviously, because of $h_n^{(1)}(k_1 r)$

$$G_s(\bar{r}, \bar{r}') = 0(r)^{-1}. \tag{45}$$

For $k_2 a \ll 1$, we can asymptotically expand the spherical Bessel functions of arguments $k_1 a$ and $k_2 a$ appearing in (44). Then

$$G_s(\bar{r}, \bar{r}') = 0(k_2 a)^3. \tag{46}$$

For $k_1 a \gg 1$, we can either utilize the well-known Watson transformation to change (44) into a new representation such that the leading term of the asymptotic expansion of the new series is enough to give an excellent asymptotic representation of (44) or simply construct the asymptotic expansion of (44) by the "geometrical theory of diffraction." The dominant term of $G_s(\bar{r}, \bar{r}')$ is the externally reflected field,

$$\frac{-1}{k_1 a} \left\{ \frac{a^2 \sin 2\Psi_3}{2rr' \sin(\Psi_1 + \Psi_2) \left[\left(1 + \frac{r'^2}{a^2} - 2\frac{r'}{a}\cos\Psi_1\right)^{1/2} \left(\frac{r^2}{a^2} - \sin^2\Psi_3\right)^{1/2} - \left(1 + \frac{r^2}{a^2} - 2\frac{r}{a}\cos\Psi_2\right)^{1/2} \left(\frac{r'^2}{a^2} - \sin^2\Psi_3\right)^{1/2} \right]} \right\}^{1/2}$$

$$\cdot \left(\frac{\alpha\cos\Psi_3 - \beta\sqrt{N^2 - \sin^2\Psi_3}}{\alpha\cos\Psi_3 + \beta\sqrt{N^2 - \sin^2\Psi_3}}\right)$$

$$\exp\{ik_1[(r^2 + a^2 - 2\,ar\cos\Psi_2)^{1/2} + (r'^2 + a^2 - 2\,ar'\cos\Psi_1)^{1/2}]\} = 0(k_1 a)^{-1}, \tag{47}$$

where the physical meaning of Ψ_1, Ψ_2, Ψ_3, r, and r' are given in Fig. 3. Hence

$$G_s(\bar{r}, \bar{r}') = 0(k_1 a)^{-1}. \tag{48}$$

We now consider the scattering of coherent waves by an acoustically

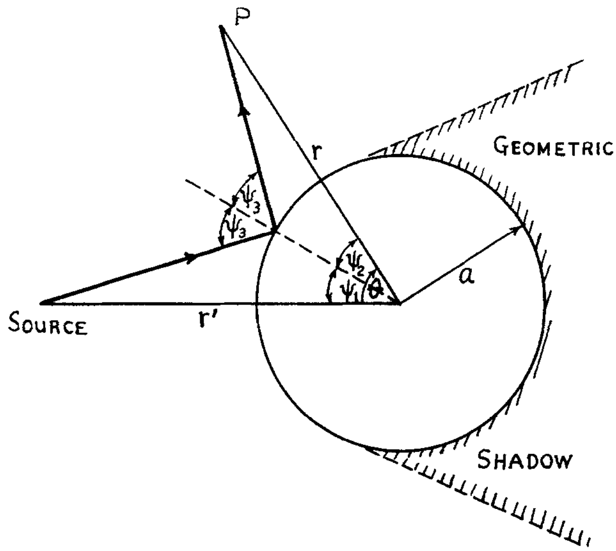

FIG. 3. The geometry of the scattering problem is shown. In addition, the physical meaning of r, r', a, ψ_1, ψ_2, and ψ_3 are given.

hard sphere imbedded in a random medium. The total field u satisfies equations (39) ($\omega(a, q) \neq 0$), (43), and

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = 0. \tag{49}$$

Upon following the same approach in the previous case, it is found that in the cases of $k_1 r \gg 1$, $k_1 a \ll 1$, and $k_1 a \gg 1$, Eqs. (45), (46), and (48), respectively, are true.

A plane wave incident upon a flat interface separating two semi-infinite media, one of the incident side being random perturbed and the other being deterministic, is a limiting case of the sphere problem ($a \to \infty$). Hence $\langle u \rangle$ satisfies (32). For the case of the lossy transparent sphere, $\langle u \rangle$ on the side of random medium is simply

$$\langle u \rangle = \exp[ik_1\tilde{n}(x \sin\bar{\theta} - y\cos\bar{\theta})] + R\exp[ik_1\tilde{n}(x\sin\bar{\theta} + y\cos\bar{\theta})] + 0(\epsilon^3), \tag{50}$$

where

$$R = \frac{\alpha k_1 \tilde{n}\cos\bar{\theta} - \beta k_2 \cos\bar{\varphi}}{\alpha k_1 \tilde{n}\cos\bar{\theta} + \beta k_2 \cos\bar{\varphi}}, \tag{51}$$

$\bar{\theta}$ and $\bar{\varphi}$ is the angle of incidence and the angle of transmission, respectively

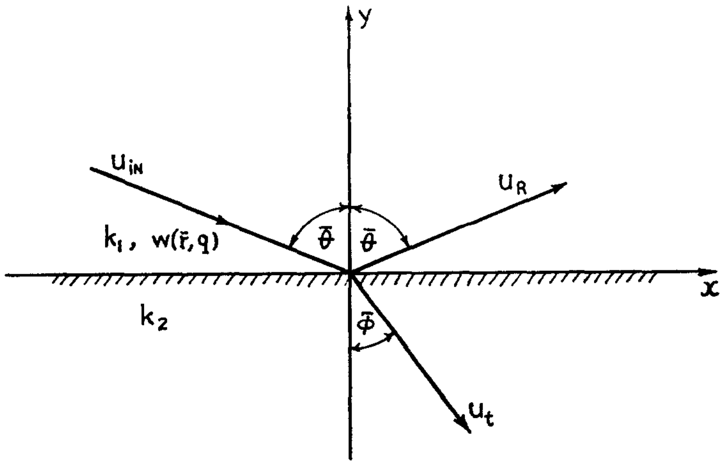

FIG. 4. The directions of propagation of incident, reflected, and transmitted waves are shown.

(Fig. 4). The reflection coefficient R can be expanded into a power series of ϵ as

$$R = R_0 + \epsilon^2 R_2 + 0(\epsilon^2), \tag{52}$$

where

$$R_0 = \frac{\alpha k_1 \cos \bar{\theta} - \beta k_2 \cos \bar{\varphi}}{\alpha k_1 \cos \bar{\theta} + \beta k_2 \cos \bar{\varphi}} \tag{53}$$

is the Fresnel reflection coefficient and

$$R_2 = \frac{\beta k_2 (1 + R_0) \left[\langle \omega^2 \rangle - i2k_1 \int_0^\infty (e^{i2k_1 \ell} - 1)\, C(\ell)\, d\ell \right] \cos \bar{\varphi}}{2(\alpha k_1 \cos \bar{\theta} + \beta k_2 \cos \bar{\varphi})} \tag{54}$$

is the correction term due to the randomness of the medium. For the case of the acoustically hard sphere, Eq. (50) is still valid except

$$R = 1. \tag{55}$$

The results in Eqs. (52), (53), (54), and (55) agree with that of Chen [14] in which a more elementary approach is adopted.

References

1. Y. M. Chen. On scattering of waves by objects imbedded in random media: Stochastic linear partial differential equations and scattering of waves by conducting sphere imbedded in random media. *J. Math. Phys.* **5** (1964), 1541-1546.
2. J. B. Keller. Stochastic Equations and wave propagation in random media. *Proc. 16th Symp. Appl. Math.* (Am. Math. Soc.) (1964), 145-170.
3. K. Huang. "Statistical Mechanics." Wiley, New York, 1963.
4. L. Liu and K. W. Wong. Bose system of hard spheres. *Phys. Rev.* **132** (1963), 1349-1353.
5. R. Courant and D. Hilbert. "Methods of Mathematical Physics," Vol. II. Wiley (Interscience), New York, 1962.
6. J. B. Keller. The geometrical theory of diffraction. *Proc. Symp. Microwave Optics*, McGill University, Montreal, Canada, 1953.
7. J. B. Keller. Geometrical theory of diffraction. *J. Optical Soc. Am.* **52** (1962), 116-130.
8. Y. M. Chen. Diffraction by a smooth transparent object. *J. Math. Phys.* **5** (1964), 820-832.
9. B. R. Levy and J. B. Keller. Diffraction by a smooth object. *Commun. Pure Appl. Math.* **12** (1959), 159-209.
10. W. C. Meecham. Space Technology Labs. Report. Los Angeles, Calif., 1961.
11. P. M. Morse and H. Feshbach. "Methods of Theoretical Physics," Vol. I. McGraw-Hill, New York, 1953.
12. V. Twersky. On scattering of waves by random distributions. I. Free-space scattering formalism. *J. Math. Phys.* **3** (1962), 700-715.
13. N. Zitron and S. N. Karp. Higher order approximations in multiple scattering. *J. Math. Phys.* **2** (1961), 394-402.